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SOLVED PYQ

Paper: Introductory Mathematical Methods for Economics

Course: B. A.(Hons.) Economics I Year

Year: 2021

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Modified as Per New NEP Syllabus

ECON002: Introductory Mathematical Methods for Economics
Course : B. A.(Hons.) Economics I Year
Duration : 3 Hours **Maximum Marks : 90**

SEMESTER-1
2021

All questions are to be attempted.

Q. 1. Answer any two parts.

(a) Solve the inequalities.

(i) $\left|1+\frac{3}{x}\right|>2$

Ans. $\left|1+\frac{3}{x}\right|>2$

$\Rightarrow \quad 1+\frac{3}{x}>2 \qquad$ or $\quad 1+\frac{3}{x}<-2$

$\Rightarrow \quad \frac{3}{x}>1 \qquad$ or $\quad \frac{3}{x}<-3$

$\Rightarrow \quad x<3 \qquad$ or $\quad \frac{x}{3}>\frac{-1}{3}$

$\Rightarrow \quad x<3 \qquad$ or $\quad x>-1$

(ii) $\frac{2}{x}<3$

Ans. $\frac{2}{x}<3 \qquad \Rightarrow \frac{x}{2}>\frac{1}{3}$

$\Rightarrow \quad x>\frac{2}{3}$

(b) (i) $f(x)=x^2+kx+1$ **for all** x**, if** $f(x)$ **is an even function, find** k**.**

Ans. $f(x)=x^2+kx+1$

$f(-x)=f(x)$ (Because f is an even function

$$\Rightarrow \quad (-x)^2+k(-x)+1 = x^2 kx+1$$
$$\Rightarrow \quad -kx = kx$$
$$\Rightarrow \quad 2kx = 0$$
$$\Rightarrow \quad k = 0$$

(ii) Find the domain and range of $\sqrt{5-4x-x^2}$**.**

Ans.
$$f(x) = \sqrt{5-4x-x^2}$$
$$= \sqrt{-\left(x^2+4x-5\right)}$$

$$= \sqrt{-\left(x^2 + 5x - x - 5\right)}$$

$$= \sqrt{-(x+5)(x-1)}$$

Domain $\quad -(x+5)(x-1) \geq 0$

$\Rightarrow \quad (x+5)(x-1) \leq 0$

+ve $\quad$ –ve $\quad$ +ve

$-\infty \quad -5 \quad 1 \quad \infty$

So, $\quad x \in [-5, 1]$

Range

Let $\quad f(x) = y$

$$y = \sqrt{5 - 4x - x^2}$$

$\Rightarrow \quad y^2 = 5 - 4x - x^2$

$\Rightarrow \quad x^2 + 4x + y^2 - 5 = 0$

$\Rightarrow$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(y^2 - 5)}}{2 \times 1}$$

$$= \frac{-4 \pm \sqrt{16 - 4(y^2 - 5)}}{2}$$

$\Rightarrow$

$$x = \frac{-2 \pm \sqrt{-y^2 + 9}}{1}$$

So, $\quad -y^2 + 9 \geq 0$

$\Rightarrow \quad 9 > y^2$

$\Rightarrow \quad -3 \leq y \leq 3$

Since y is positive

Hence $\quad y \in [0, 3]$

(c) Consider the proposition $2x + 5 \geq 13$

(i) Is the condition $x > 0$ necessary, or both necessary and sufficient for the proposition to be satisfied?

Ans. $\quad 2x + 5 \geq 13$

$\Rightarrow \quad 2x > 8$

$\Rightarrow \quad x > 4$

So, $x > 0$ is just a necessary condition because suppose even if $x < 0$, it may or may not be greater than 4 (for example $x = 2$).

(ii) Answer (i) if $x \geq 2$ is replaced by $x \geq 4$. **3.5, 3.5**

Ans. It is both necessary and sufficient now.

Q. 2. Attempt any four of the following :

(a) Find the limit of the following:

(i) $\lim_{x \to 0} \dfrac{\sqrt{x+3} - \sqrt{3}}{x}$

Ans. $\lim\limits_{x\to 0}\dfrac{\sqrt{x+3}-\sqrt{3}}{x}$

$$\Rightarrow \quad \lim_{x\to 0}\frac{\sqrt{x+3}-\sqrt{3}}{x}\cdot\frac{\sqrt{x+3}+\sqrt{3}}{\sqrt{x+3}+\sqrt{3}}$$

$$\Rightarrow \quad \lim_{x\to 0}\frac{x+3-3}{x}\cdot\frac{1}{\sqrt{x+3}+\sqrt{3}}$$

$$\Rightarrow \quad \lim_{x\to 0}\frac{1}{\sqrt{x+3}+\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

(ii) $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+5}}{3x^2-2}$

Ans. $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+5}}{\sqrt{3x^2-2}}$

$$\lim_{x\to 0}\frac{\sqrt{x^2+5}}{x^4}$$

$$\Rightarrow \quad \lim_{x\to 0}\frac{\dfrac{\sqrt{x^2+5}}{x^4}}{\dfrac{3x^4-2}{x^4}}$$

$$\Rightarrow \quad \lim_{x\to 0}\frac{\sqrt{1+\dfrac{5}{x^2}}}{3-\dfrac{2}{x^4}}$$

$$\Rightarrow \quad \frac{1}{3}$$

(b) Find the vertical, horizontal and oblique (if any) asymptote of the $\dfrac{(x-1)^3}{x^2}$.

Ans. Let $$y = \frac{(x-1)^3}{x^2}$$

Vertical asymptote

$$x^2 = 0 \Rightarrow x = 0$$

Horizontal asymptote

Since the degree of numerator is higher than the degree of denominator, there is no horizontal asymptote.

Oblique asymptote

$$y = \frac{x^3 - 1^3 - 3(x)^2(1) + 3(x)(1)^2}{x^2}$$

$$\Rightarrow \quad y = \frac{x^3 - 1 - 3x^2 + 3x}{x^2}$$

$$= \frac{x^3 - 3x^2 + 3x - 1}{x^2}$$

$$= (x-3) + \frac{3}{x} - \frac{1}{x^2}$$

So the oblique asymptote is $y = x - 3$.

(c) Find the points of discontinuity (if any) of the following functions:

(i)
$$f(x) = \begin{cases} x+1, & x \geq 2 \\ 2x-1, & 1 < x < 2 \\ x-1, & x \leq 1 \end{cases}$$

Ans.
$$f(x) = \begin{cases} x+1 & x \geq 2 \\ 2x-1 & 1 < x < 2 \\ x-1 & x \leq 1 \end{cases}$$

Checking for continuity at $x = 2$

$$\lim_{x \to 2^-} (2x-1) \Rightarrow \lim_{n \to 0} (2(2-h)-1)$$
$$= 3$$
$$\lim_{x \to 2^+} (x+1) \Rightarrow \lim_{n \to 0} (2+h+1)$$
$$= 3$$
$$f(2) = 2+1$$
$$= 3$$

So, f is continuous at $x = 2$

Checking continuity at $x = 1$

$$\lim_{x \to 1^-} x-1 \Rightarrow \lim_{n \to 0} (2(1-h)-1) = 0$$

$$\lim_{x \to 1^+} (2x-1) \Rightarrow \lim_{n \to 0} (2(1+h)-1) = 1$$

So, f is continuous everywhere except at x = 1.

(ii)
$$f(x) = \begin{Bmatrix} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{Bmatrix}$$

Ans.
$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$$

This function is continuous everywhere.

(d) Examine the convergence of the following sequences :

(i) $\left\{(-1)^{n-1}\frac{1}{n}\right\}$

Ans. $\left\{(-1)^{n-1}\frac{1}{n}\right\}$

$$\frac{1}{n} \text{ as } n \to \infty = 0$$

Hence the equence converges.

(ii) $\left\{\frac{x^2+1}{x^2+2}\right\}$

Ans. $\left\{\frac{x^2+1}{x^2+2}\right\}$

$$\lim_{x\to x}\frac{x^2+1}{x^2+2}$$

$$\Rightarrow \quad \lim_{x\to\infty}\frac{1+\frac{1}{x^2}}{1+\frac{2}{x^2}}$$

$$\Rightarrow \quad 1$$

Hence the sequence converges.

(e) What is the present value of 15 annual deposits of ₹ 3000 each when the first deposit is 1 year from now and the interest rate is 12%.

Ans. Present Value of the deposits

$$= \frac{3000}{(1.12)}+\frac{3000}{(1.12)^2}+\ldots+\frac{3000}{(1.12)^{15}}$$

$$= \frac{3000}{(1.12)}\left[1+\frac{1}{(1.12)}+\ldots+\frac{1}{(1.12)^{11}}\right]$$

$$= \frac{3000}{(1.12)}\left[\frac{1-\left(\frac{1}{1.12}\right)^{15}}{1-\frac{1}{(1.12)}}\right]$$

$$= ₹\ 18{,}011.99$$

(f) The expenditure of house hold on consumer goods (C) is related to the household income (Y) in the following way. When the household's income is ₹ 1000 the expenditure on consumer goods is ₹ 900, and whenever income is increased by ₹ 100, the expenditure on consumer

goods is increased by ₹ 80. Express the expenditure on consumer goods a function of income, assuming a linear relationship. 4,4,4,4

Ans. Let $C = aY + b$

When $Y = 1000, C = 900$

$$\frac{\Delta C}{\Delta Y} = \frac{80}{100} = a$$

$\Rightarrow$ So, $a = 0.8$

and $900 = 0.8 \times 1000 + b$

$\Rightarrow$ $b = 100$

So, $C = 0.8Y + 100.$

Q. 3. Attempt any three of the following :

(a) Prove that $(1 + x)^m$ is approximately equal to $1 + mx$ for x close to 0 and use this to find approximation to the following:

Ans. $(1 + x)^m = 1 + mx + m(m-1)\frac{x^2}{2!} + \ldots$ [Using Taylor approximation]

When $x \to 0$, terms three onwards will be very small in value.

Hence $(1 + x) - 1 + mx$

(i) $\sqrt{37} = \sqrt{36+1}$

Ans. $\sqrt{37} = (36 + 1)$

Let $f(x) = \sqrt{x}$

So, $f(36) = 6$

Now, $f'(x) = \frac{1}{2\sqrt{x}}$

So, $df(x) \approx f'(x)\Delta x$

$$\approx \frac{1}{2\sqrt{36}} \cdot 1$$

$$\approx \frac{1}{12}$$

So, $f(37) \approx 6 + \frac{1}{12} \approx 6.083$

(ii) $\sqrt[3]{33}$

Ans. $\sqrt[5]{33} = \sqrt[5]{32+1}$

Let $f(x) = \sqrt[5]{x}$

So, $f(32) = 2$

$$f'(x) = \frac{1}{5x^{4/5}} \Rightarrow f''(32) = \frac{1}{5-16} = \frac{1}{80}$$

So, $$f(33) = 2+\frac{1}{80}$$

$$= 2.0125$$

Prove

(b) Prove that :

(i) $El_x(f-g) = \dfrac{fEl_x f - gEl_x g}{f-g}$

Ans.
$$El_x(f-g) = \frac{d(f-g)}{dx}\cdot\frac{x}{(f-g)}$$

$$= \left[\frac{df}{dx}\cdot x - \frac{dg}{dx}\cdot x\right]\frac{1}{(f-g)}$$

$$= \left[\frac{df}{dx}\cdot\frac{x}{f}\cdot f - \frac{dg}{dx}\cdot\frac{x}{g}\cdot g\right]\left(\frac{1}{f-g}\right)$$

$$= \frac{f\cdot El_x f - gEl_x g}{f-g}$$

(iii) If $\frac{y}{x} = \ln(xy)$ find the elasticity of y with respect to x.

Ans. $\dfrac{y}{x} = \ln x + \ln y$

$$\Rightarrow \quad y = x\ln x + x\ln y$$

$$\Rightarrow \quad \frac{dy}{dx} = x\cdot\frac{1}{x} + \ln x + x\cdot\frac{1}{y}\cdot\frac{dy}{dx} + \ln y$$

$$\Rightarrow \quad \frac{dy}{dx}\left(1-\frac{x}{y}\right) = 1 + \ln(xy)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1+\ln(xy)}{1-\frac{x}{y}}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y(1+\ln xy)}{y-x}$$

$$El_x y = \frac{dy}{dx}\cdot\frac{x}{y}$$

$$= \frac{x + x\ln(xy)}{y-x}$$

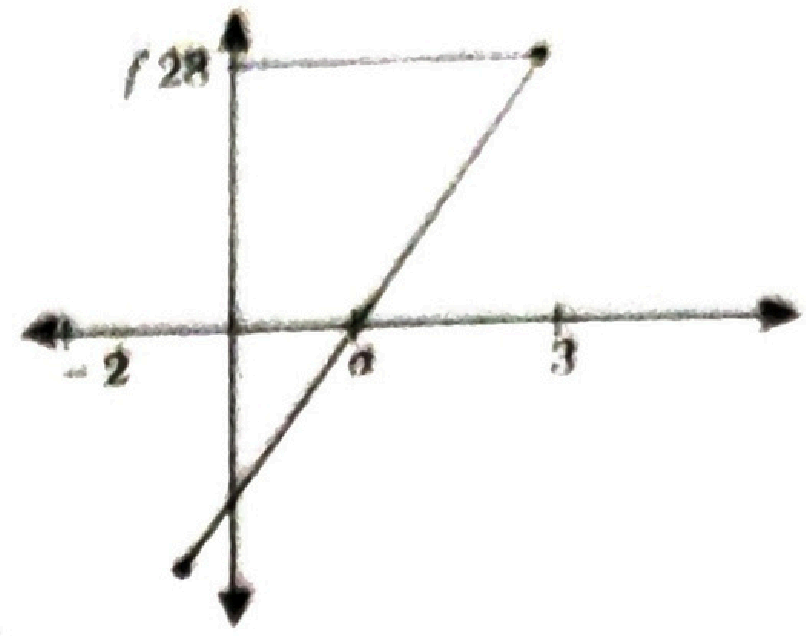
(c) Prove that each of the following equations have at least one solution in the given interval.

(i) $x^3 + 3x - 8 = 0\ [-2, 3]$

Ans. $x^3 + 3x - 8 = 0$

Let

$$f(x) = x^3 + 3x - 8$$
$$f(-2) = -8 - 6 - 8$$
$$= -22$$
$$f(3) = 27 + 3(3) - 8$$
$$= 28$$



A rough graph of f is

So, there is atleast one point a where f is zero.

(ii) $\sqrt{x^2+1} = 3x$ **in (0, 1)**

Ans. $\sqrt{x^2+1} - 3x = 0$

Let

$$f(x) = \sqrt{x^2+1} - 3x$$
$$f(0) = 1 - 3(0)$$
$$= 1$$
$$f(1) = \sqrt{2} - 3$$
$$= -1.5857$$

Since the value of the function changes from positive to negative, it must cross the x-axis, atlest once.

(d) Let f be defined on [0, 1] by $f(x) = 2x^2 - x^4$.

(i) Find the range of the function

Ans.

$$f(0) = 0$$
$$f(1) = 2 - 1$$
$$= 1$$
$$\therefore \quad f(x) \in [0, 1]$$

(ii) Show that f has an inverse function g, and find a formula for g.

5,5,5

Ans. $f(x) = 2x^2 - x^4$

Let $f(x) = y$

So, $2x^2 - x^4 = y$

$\Rightarrow$ $x^4 - 2x^2 + y = 0$

Let $x^2 = z$

So, $z^2 - 2z + y = 0$

$$\Rightarrow \quad z = \frac{(-2) \pm \sqrt{(-2)^2 - 4(y)}}{2 \times 1}$$

$$\Rightarrow \quad z = \frac{2 \pm \sqrt{4 - 4y}}{2}$$

$$z = \frac{1 \pm \sqrt{1 - y}}{2}$$

Since x can not be negative

$$z = \frac{1 \pm \sqrt{1-y}}{2}$$

$$\Rightarrow \quad x^2 = \frac{1+\sqrt{1-y}}{2}$$

$$\Rightarrow \quad x = \sqrt{\frac{1+\sqrt{1-y}}{2}}$$

Q. 4. Answer any three parts.

(i) Find the formula for n^{th} derivative of $y = \frac{1}{y}(1-x)$

Ans.

$$y = \frac{1}{x}(1-x) = \frac{1}{x} - 1$$

$$\frac{dy}{dx} = \frac{-1}{x^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{(-1)(-2)}{x^3} = \frac{2!}{x^3}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{(-1)(-2)(-3)}{x^4} = \frac{(-1)3!}{x^4}$$

So,

$$\frac{d^ny}{dx^n} = \frac{(-1)^n n!}{x^{n+1}}$$

Or

If $x^3 - y^3 = 1$ find $\frac{d^2y}{dx^2}$.

$$x^3 - y^3 = 1$$

$$\Rightarrow \quad 3x^2 - 3y^2 \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{x^2}{y^2} = \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{y^2 \cdot 2x - x^2 \cdot 2y \cdot dy/dx}{y^4}$$

$$= \frac{2xy^2 - \frac{2x^4}{y}}{y^4}$$

$$= \frac{2xy^3 - 2x^4}{y^5}$$

(b) The price of agricultural goods is going up by 4 percent each year and quantity by 2 percent. What is the annual growth of revenue (R) derived from the agricutural sector?

Ans. Increasing in price = 4% . (p)

Increase in quantity = 2% . (x)

Revenue = px

$\Rightarrow$ $R = px$

$\Rightarrow$ $\text{Log } R = \log p + \log x$

$\Delta\log R = \Delta\log p + \Delta\log x$

$\Rightarrow$ % increase in Revenue = 6%

(c) Show that the tangent at the point (1, 1) on the rectangular phyperbola $xy = 1$ cuts equal lengths of the axes.

Ans.

$$xy = 1$$

$$\Rightarrow \quad x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-y}{x}$$

$$\Rightarrow \quad \left[\frac{dy}{dx}\right]_{(1,1)} = -1$$

So, the equation of the tangent

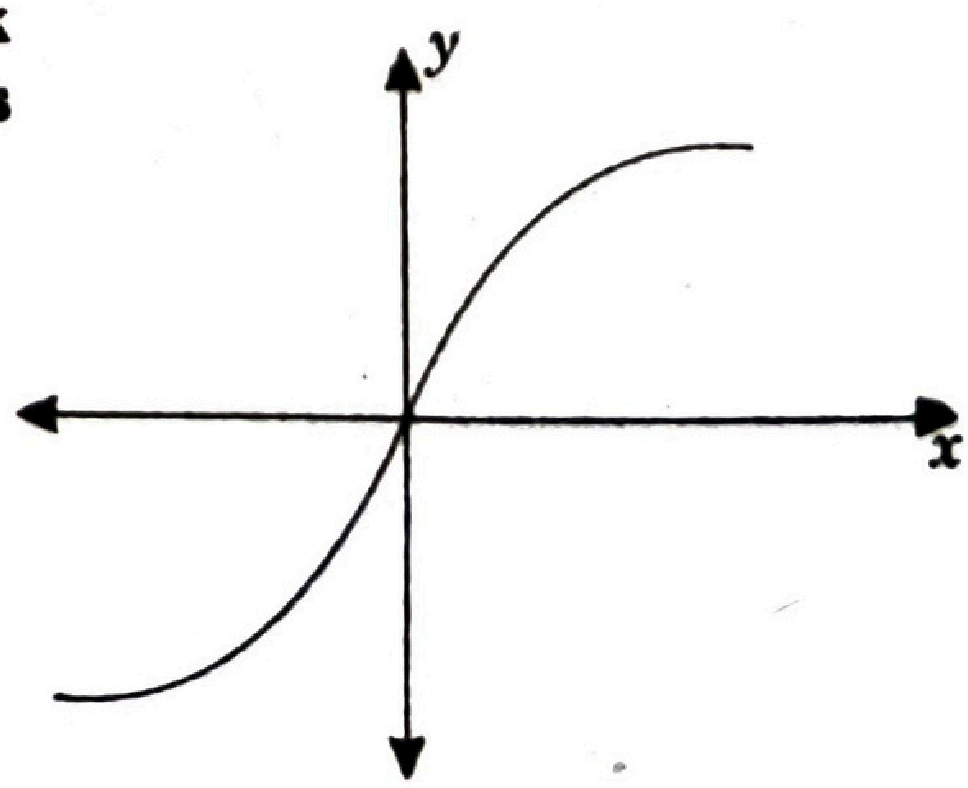
$$\frac{y-1}{x-1} = -1$$

$$\Rightarrow \quad y - 1 = -x + 1$$

$$\Rightarrow \quad x + y = 2$$

So, the line cuts the axes at 1 each.

(d) Show that the curve $y = \sqrt[3]{x}$ is convex from below for negative values of x and concave from below for positive values of x. Deduce that origin is point of inflection. Why is the point not given by the criterion that the second derivatives is zero? Check the result by considering the function as the inverse of $y = x^3$.



Ans.

$$y = x^{1/3}$$

$$\frac{dy}{dx} = \frac{1}{3}x^{\frac{-2}{3}}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-2}{9}x^{\frac{-5}{3}}$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2} = \frac{-2}{9x^{\frac{5}{3}}}$$

The second.

(e) Find the absolute maximum and minimum of $f(x) = 4x^3 - 8x^2 + 1$ on the closed interval $[-1, 1]$.

Ans.

$$f(x) = 4x^3 - 8x^2 + 1$$

$$\Rightarrow \qquad f'(x) = 12x^2 - 16x = 0$$

$$\Rightarrow \qquad 4x(3x - 4) = 0$$

$$\Rightarrow \qquad x = 0, \; x = \frac{4}{3}$$

$$f(-1) = -4 - 8 + 1$$

$$= -11$$

$$f(1) = -3$$

$$f(0) = 1$$

$$f\left(\frac{4}{3}\right) = 4\left(\frac{4}{3}\right)^3 - 8\left(\frac{4}{3}\right)^2 + 1$$

$$= \frac{256}{27} - \frac{128}{9} + 1$$

$$= \frac{283 - 384}{27}$$

$$= \frac{-101}{27}$$

So, absolute minimum is $\frac{-101}{27}$ and absolute maximum is 1.

Q. 5. Answer any three parts.

(a) The art collection of a decreased painter has an estimated value of $V = 200{,}000(1.25)^{\sqrt[3]{t^2}}$. How long should the excutioner of estate hold the collections before putting it up for sale if the discount rate under continuous compounding is 6 percent?

Ans. $V = 200000(1.25)^{t^{2/3}}$

Discounted value of the art collection at time period 't'

$$= 200000(1.25)^{t^{2/3}} \cdot e^{-0.06t}$$

$$\Rightarrow \qquad D.V_t = 200{,}000(1.25)^{t^{2/3}} e^{-0.06t}$$

$$\log D.V_t = \log 2{,}00{,}000 + t^{2/3} \log(1.25) - 0.06t$$

$$\Rightarrow \quad \frac{d \log D.V_t}{dt} = \frac{2}{3}t^{-1/3}\log(1.25) - 0.06$$

$$\Rightarrow \quad \frac{d^2 \log D.V_t}{dt^2} = \frac{-1}{9}t^{-4/3}\log(1.25)$$

For maximum, $\frac{d \log DV_t}{dt} = 0$

$$\Rightarrow \quad \frac{2}{3}t^{-1/3}\log(1.25) = 0.06$$

$$\Rightarrow \quad \frac{2}{3}t^{-1/3} = \frac{0.06}{\log(1.25)}$$

$$\Rightarrow \quad \frac{2\log(1.25)}{3 \times 0.06} = t^{1/3}$$

$$t^* = \left(\frac{\log(1.25)}{0.09}\right)^3$$

$$t^* = 15.24$$

$$\frac{d^2 \log D.V_t}{dt^2} < 0 \text{ for } t = 15.24.$$

Hence he should hold the collection for 15.24 time period.

(b) $D = a - b\,(p + t)$, $S = C + dp$ where a, b, c, d are positive constants and p is determined when $D = S$. p is implicity defined as a function of t (unit tax).

(i) Find $\frac{dp}{dt}$ and sign of $\frac{dp}{dt}$.

(ii) Compute the tax revenue T as a function of t. For what value of t does the quadratic function T reach maximum.

Ans.

$$D = S$$

$$\Rightarrow \quad a - b\,(p + t) = C + dp$$

$$\Rightarrow \quad a - bp - bt = C + dp$$

$$\Rightarrow \quad \frac{a - bt}{b + d} = p$$

(i) $$\frac{dp}{dt} = \frac{-d}{b+d} < 0$$

(ii) Tax revenue $= t \,.\, x$

$$= t \cdot \left(C + d\frac{(a - bt)}{b + d}\right)$$

$$\text{T.K.} = t\left(\frac{Cb + cd + ad + bdt}{b+d}\right)$$

$$\Rightarrow \quad \frac{d\text{TR}}{dt} = t\left(\frac{-bd}{b+d}\right) + \left(\frac{Cb + Cd + ad - bdt}{b+d}\right)$$

$$\Rightarrow \quad \frac{d^2\text{TR}}{dt^2} = \frac{-bd}{b+d}$$

For maxima,
$$\frac{d\text{TR}}{dt} = 0$$

$$\Rightarrow \quad \frac{Cb + Cd + ad}{b+d} = \frac{2bdt}{b+d}$$

$$\Rightarrow \quad t^* = \frac{Cb + Cd + ad}{2bd}$$

$$\frac{d^2\text{TR}}{dt^2} < 0 \text{ for } \quad t^* = \frac{Cb + Cd + ad}{2bd}$$

$$t^* = \frac{Cb + cd + ad}{2bd}$$, **tax revenue is maxima**

(c) $f(x) = 5x^4 - x^5$. For what value of x is $f(x)$ concave upward and concave downward. Find all the points of inflection.

Ans.
$$f(x) = 5x^4 - x^5$$
$$\Rightarrow \quad f'(x) = 20x^3 - 5x^4$$
$$\Rightarrow \quad f''(x) = 60x^2 - 20x^3$$

Concave upward if $f''(x) > 0$
$$\Rightarrow \quad 60x^2 - 20x^2 > 0$$
$$\Rightarrow \quad 20x^3 (3 - x) > 0$$

So, $x < 3$

Concave downward if $f''(x) < 0$
$$\Rightarrow \quad x > 3$$

Point of inflection if $f''(x) = 0$
$$x = 3 \quad \text{or } x = 0$$

(d) Find the intervals where the following cubic cost function is convex and where it is concave, and find the unique inflection point $C(Q) = a(Q^3 - bQ^2 + cQ + d$ $(a > 0, b < 0, c > 0, d > 0)$ **5,5,5**

Ans.
$$C(Q) = aQ^3 - bQ^2 CQ + d$$
$$\Rightarrow \quad C'Q = 3aQ^2 - 2bQ + C$$
$$\Rightarrow \quad C''(Q) = 6aQ - 2b$$

Intervals on which the function is concave
$$C''(Q) < 0$$

$$\Rightarrow \qquad 6aQ - 2b < 0$$

$$\Rightarrow \qquad Q < \frac{2b}{6a}$$

$$\Rightarrow \qquad Q < \frac{b}{3a}$$

Interval on which the function is convex

$$C''(Q) > 0$$

$$\Rightarrow \qquad Q > \frac{b}{3a}$$

For inflection, $\quad C''(Q) = 0$

$$\Rightarrow \qquad Q = \frac{b}{3a}$$

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